Homework 3

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Be first to report a math error for extra credit.

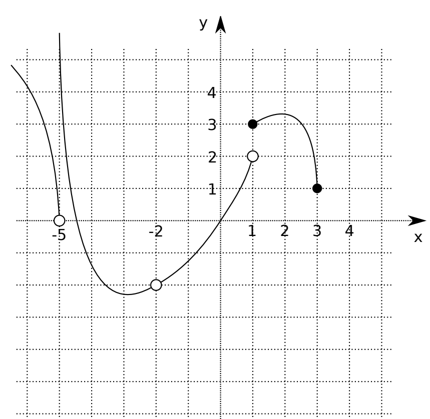
Read Stewart sections 2.1-2.5. Alternatively, read
`http://en.wikibooks.org/wiki/Calculus/Limits`.
Both sources contain ample selections of practice exercises.

Problems with stars [*] are optional. You do not have to solve them, and no quiz or test will include anything like them.

Exercise 1. State the definition of the limit (either formal or informal), i.e. define what it means when we say that $\lim_{x\to a} f(x) = L$.

Exercise 2. The graph of $f(x)$ is shown below. Find the following limits if they exist.

(1) $\lim_{x\to -5} f(x)$,
(2) $\lim_{x\to -5^+} f(x)$,
(3) $\lim_{x\to -5^-} f(x)$,
(4) $\lim_{x\to -2} f(x)$,
(5) $\lim_{x\to 0} f(x)$,
(6) $\lim_{x\to 1^+} f(x)$,
(7) $\lim_{x\to 1^-} f(x)$,
(8) $\lim_{x\to 1} f(x)$.



In the following 2 exercises, plot functions with given properties.

Exercise 3. $f(3) = 2$, $\lim_{x\to 3^+} f(x) = 1$, $\lim_{x\to 3^-} f(x) = -1$.

Exercise 4. $f(0) = \lim_{x\to 0^+} f(x) = 1$, f is defined on an open interval containing 0, $\lim_{x\to 0^-} f(x)$ does not exist.

In the following 10 exercises, prove the given statement or find the given limit, if it exists.

Exercise 5. $\lim_{x\to 5} \dfrac{x^2-6x+5}{x-5} = 4.$

Exercise 6. $\lim_{x\to 5} \dfrac{x^2-5x+6}{x-5}$ does not exist.

Exercise 7. $\lim_{t\to -3} \dfrac{t^2-9}{2t^2+7t+3} = \dfrac{6}{5}.$

Exercise 8. $\lim_{h\to 0} \dfrac{(4+h)^2-16}{h} = 8.$

Exercise 9. $\lim_{x\to -2} \dfrac{x+2}{x^3+8} = \dfrac{1}{12}.$

Exercise 10. $\lim_{x\to -4} \dfrac{\frac{1}{4}+\frac{1}{x}}{4+x} = -\dfrac{1}{16}.$

Exercise 11. $\lim_{x\to 16} \dfrac{4-\sqrt{x}}{16x-x^2} = \dfrac{1}{128}.$

Exercise 12. $\lim_{t\to 0} \left(\dfrac{1}{t\sqrt{1+t^2}} - \dfrac{1}{t} \right).$

Exercise 13. $\lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}.$

Exercise 14. $\lim_{x\to -4} \dfrac{\sqrt{x^2+9}-5}{x+4}.$

Exercise 15. Show that $\lim_{x\to 0}(x^2\cos(\pi x)) = 0$.

Exercise 16. Show that $\lim_{x\to 0^+} \sqrt{x}e^{\sin(\pi/x)} = 0$.

Exercise 17. Find $\lim_{x\to 1} g(x)$ if $2x \leqslant g(x) \leqslant x^4 - x^2 + 2$ for all x.

In the following 3 exercises, find the limit if it exists. If it does not, explain why.

Exercise 18. $\lim_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|} \right).$

Exercise 19. $\lim_{x\to -6} \dfrac{2x+12}{|x+6|}$.

Exercise 20. $\lim_{x\to -2} \dfrac{2-|x|}{2+x}$.

Exercise 21. Let $f(x) = \lfloor \sin(x) \rfloor$ with $-\pi \leqslant x \leqslant \pi$, where $\lfloor x \rfloor$ denotes the largest integer $n \leqslant x$ (this function is known as the *floor function*). Sketch the graph of $f(x)$. Evaluate each limit or explain why it doesn't exist.

(1) $\lim_{x\to 0} f(x)$.
(2) $\lim_{x\to \pi/2^+} f(x)$.
(3) $\lim_{x\to \pi/2^-} f(x)$.
(4) $\lim_{x\to \pi/2} f(x)$.

Exercise 22. According to Wikipedia, *length contraction* is the physical phenomenon of a decrease in length detected by an observer in objects that travel at any non-zero velocity relative to that observer. It is a relativistic effect which is only noticable at a substantial fraction of the speed of light. As the magnitude of the velocity approaches the speed of light, the effect becomes dominant, as can be seen from the formula

$$L = L_0\sqrt{1 - v^2/c^2},$$

where L_0 is the proper length of an object, L is the observed length, v is the relative velocity of the moving object, and c is the speed of light.

Find $\lim_{v\to c^-} L(v)$.

Exercise 23. [*] Use the formal definition of the limit to show that

$$\lim_{x\to a} \frac{1}{x} = \frac{1}{a}.$$

Exercise 24. Define what it means for a function $f(x)$ to be continuous at a; to be continuous on an interval (a, b).

Exercise 25. Let $f(x)$ be defined as in the exercise (2). Find all the intervals on which $f(x)$ is continuous.

Exercise 26. Locate the discontinuities of the function $y(x) = \dfrac{1}{1+e^{1/x}}$. Plot the function.

Exercise 27. Show that the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geqslant 1 \end{cases}$$

is continuous on $(-\infty, \infty)$.

Exercise 28. For what value of the constant c is the function

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

continuous on $\mathbb{R}$?

Exercise 29. Use the Intermediate Value Theorem to show that there is at least one root of the equation

$$e^x = 3 - 2x$$

in the interval $(0, 1)$.

Exercise 30. Prove that the equation

$$\cos x = x^3$$

has at least one real root.

In the following 3 exercises, sketch the graph of a function which satisfies all of the given conditions.

Exercise 31. $\lim_{x\to 0} = -\infty$, $\lim_{x\to -\infty} = 5$, $\lim_{x\to\infty} = -5$.

Exercise 32. $\lim_{x\to 2} = -\infty$, $\lim_{x\to\infty} = \infty$, $\lim_{x\to -\infty} = 0$, $\lim_{x\to 0^+} = \infty$, $\lim_{x\to 0^-} = 1$, $f(0) = 0$.

Exercise 33. $\lim_{x\to 3} = -\infty$, $\lim_{x\to\infty} = 2$, $f(0) = 0$, f is even.

In the following 11 exercises find the given limit or prove the given statement.

Exercise 34. $\displaystyle\lim_{x\to 1} \frac{2 - x}{(x-1)^2} = \infty$.

Exercise 35. $\displaystyle\lim_{x\to 2^+} e^{3/(2-x)} = 0$.

Exercise 36. $\displaystyle\lim_{x\to 3^+} \ln(x^2 - 9) = -\infty$.

Exercise 37. $\displaystyle\lim_{x\to 2\pi^-} x \csc x = -\infty$.

Exercise 38. $\displaystyle\lim_{u\to\infty} \frac{4u^4 + 5}{(u^2 - 2)(2u^2 - 1)} = 2$.

Exercise 39. $\displaystyle\lim_{x\to\infty} (\sqrt{9x^2 + x} - 3x) = \frac{1}{6}$.

Exercise 40. $\displaystyle\lim_{x\to\infty} (e^{-2x} \cos x)$.

Exercise 41. $\lim\limits_{x\to 2^-} \dfrac{x^2 - 2x}{x^2 - 4x + 4}$.

Exercise 42. $\lim\limits_{x\to \infty} \dfrac{x + 2}{\sqrt{9x^2 + 1}}$.

Exercise 43. $\lim\limits_{x\to \infty} \dfrac{\sin^2 x}{x^2}$.

Exercise 44. $\lim\limits_{x\to \pi/2^+} e^{\tan x}$.

In the following 3 exercises, plot the given curve and indicate all of its asymptotes.

Exercise 45. $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$.

Exercise 46. $y = \dfrac{x^3 - x}{x^2 - 6x + 5}$.

Exercise 47. $y = \dfrac{2e^x}{e^x - 5}$.