CHAPTER 9

Quadratic Equations

1. Complex Numbers

1.1. Imaginary Unit. Complex numbers were initially designed to assist in factoring polynomials around 1545 by Gerolamo Cardano. It was observed that an equation such as

$$x^2 = -1$$

which is provably without a real solution, can be postulated to have a solution which is not a real number, and nothing goes wrong as a result. In fact, one could say the situation improves somewhat. It becomes possible to construct a set of *complex numbers*, which includes all real numbers as well as all numbers which happen to be solutions of polynomial equations with real or complex coefficients. All of the real number axioms hold true for the complex numbers, so most theorems presented in this text hold for the complex numbers almost word for word.

DEFINITION 1.1.1 (Imaginary Unit). The solutions of the equation

$$x^2 = -1$$

are complex numbers i and its opposite $-i$:

$$i^2 = (-i)^2 = -1$$

i is usually called the *imaginary unit*, and it happens to be the principal square root of the negative unit:

$$\sqrt{-1} = i$$

THEOREM 1.1.1. If x is a positive real number, then the principal square root of $-x$ is

$$\sqrt{-x} = \sqrt{x} \cdot i$$

EXAMPLE 1.1.1. Simplify the expression $\sqrt{-100}$

SOLUTION:

$$\sqrt{-100} = \sqrt{100} \cdot i = 10i$$

ANSWER: $10i$

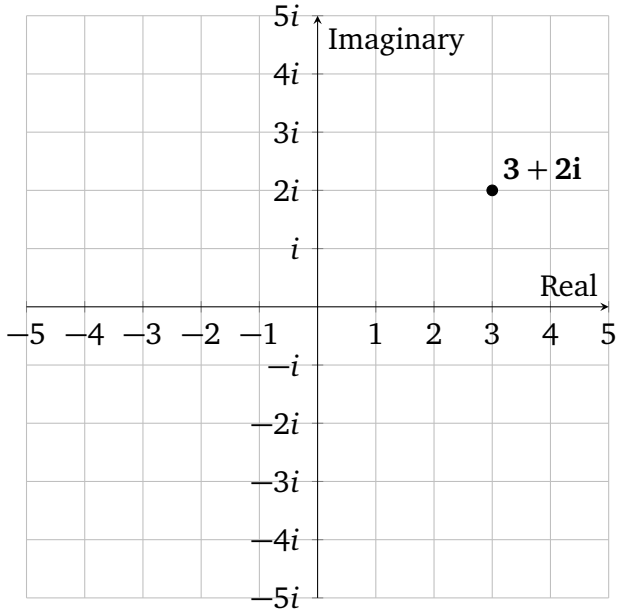
EXAMPLE 1.1.2. Simplify the expression $\sqrt{-20}$

SOLUTION:

$$\sqrt{-20} = \sqrt{20} \cdot i = 2\sqrt{5} \cdot i$$

ANSWER: $2\sqrt{5} \cdot i$

1.2. The Set of Complex Numbers. Just like the set of real numbers can be visualized as a number line, the set of complex numbers can be visualized as a number plane.



This plane works a lot like the Cartesian plane, but the coordinates are interpreted in a particular way. Every point on this plane represents a distinct complex number. The points on the real axis are the familiar real numbers. The imaginary unit i is located one unit above the origin, and the points on the imaginary axis are the real multiples of i. Every other point with coordinates (a, b), where a and b are real numbers, is a complex number which can be written as $a + bi$.

DEFINITION 1.2.1. A *complex number* is any number z which can be written in the form

$$a + bi$$

where a is a real number called the *real part of* z, b is a real number called the *imaginary part of* z, and i is the imaginary unit.

1.3. Adding and Multiplying Complex Numbers.

EXAMPLE 1.3.1. Simplify the expression $(3+5i)-(2+7i)$

SOLUTION: We treat i as if it was a variable, and combine the like terms:

$$\begin{aligned}(3+5i)-(2+7i) &= 3+5i-2-7i && \text{removed parentheses}\\ &= 1-2i && \text{combined like terms}\end{aligned}$$

ANSWER: $1-2i$

EXAMPLE 1.3.2. Simplify the expression $(1+6i)(2-i)$

SOLUTION: We treat i as if it was a variable, multiply the binomials, and combine the like terms:

$$\begin{aligned}(1+6i)(2-i) &= 2-1i+12i-6i^2\\ &= 2+11i-6i^2 && i^2 \text{ can be replaced by } -1\\ &= 2+11i-6(-1)\\ &= 2+11i+6\\ &= 8+11i && \text{combined like terms}\end{aligned}$$

ANSWER: $8+11i$

EXAMPLE 1.3.3. Simplify the expression $\dfrac{8+\sqrt{-16}}{4}$

SOLUTION: We simplify the radicand, and then cancel common factors:

$$\begin{aligned}\frac{8+\sqrt{-16}}{4} &= \frac{8+4i}{4} && \text{rewrite in terms of } i\\ &= \frac{4(2+i)}{4} && \text{factored out GCF}\\ &= 2+i && \text{canceled common factor 4}\end{aligned}$$

ANSWER: $2+i$

EXAMPLE 1.3.4. Simplify the expression $\dfrac{20-\sqrt{-50}}{10}$

SOLUTION:

$$\begin{aligned}
\frac{20-\sqrt{-50}}{10} &= \frac{20-\sqrt{50}\cdot i}{10} && \text{rewrite in terms of } i\\
&= \frac{20-\sqrt{25\cdot 2}\cdot i}{10} && \text{simplify the radicand}\\
&= \frac{20-\sqrt{25}\sqrt{2}\cdot i}{10} && \text{distributivity of } \sqrt{} \text{ over } \cdot\\
&= \frac{20-5\sqrt{2}\cdot i}{10} && \sqrt{25}=5\\
&= \frac{5(4-\sqrt{2}\cdot i)}{10} && \text{factored out GCF}\\
&= \frac{4-\sqrt{2}\cdot i}{2} && \text{canceled common factor 5}
\end{aligned}$$

If desired, this expression could be stated in the form $a+bi$ by rewriting the fraction as a sum:

$$\begin{aligned}
\frac{4-\sqrt{2}\cdot i}{2} &= \frac{4}{2}-\frac{\sqrt{2}}{2}i\\
&= 2-\frac{\sqrt{2}}{2}i
\end{aligned}$$

But we will be content with simplified fractions for answers:

ANSWER: $\dfrac{4-\sqrt{2}\cdot i}{2}$

Homework 9.1.

Simplify the given expression and state the answer in the form bi, where b is real.

1. $\sqrt{-5}$

2. $\sqrt{-6}$

3. $\sqrt{-49}$

4. $\sqrt{-81}$

5. $\sqrt{-45}$

6. $\sqrt{-18}$

7. $\sqrt{-120}$

8. $\sqrt{-160}$

Simplify the given expression.

9. $(7+3i)+(14+2i)$

10. $(12-4i)+(-1+6i)$

11. $(21-8i)-(5+5i)$

12. $(-10-i)-(2-4i)$

13. $(2-3i)-(9-3i)$

14. $(6-11i)+(17i-6)$

15. $(1+i)(1-2i)$

16. $(3-i)(4+i)$

17. $(10-3i)(2i)$

18. $(-5i)(2+4i)$

19. $30(-2+0.5i)$

20. $\frac{2}{3}(24-30i)$

21. $(7+5i)(7-5i)$

22. $(3-6i)(3+6i)$

23. $(4+i)^2$

24. $(1-3i)^2$

25. $\dfrac{-6+\sqrt{-4}}{3}$

26. $\dfrac{15-\sqrt{-9}}{5}$

27. $\dfrac{18+\sqrt{-8}}{-2}$

28. $\dfrac{-10+\sqrt{-25}}{-10}$

29. $\dfrac{-24-\sqrt{-27}}{-6}$

30. $\dfrac{-4+\sqrt{-28}}{6}$

Homework 9.1 Answers.

1. $\sqrt{5} \cdot i$

3. $7i$

5. $3\sqrt{5} \cdot i$

7. $2\sqrt{30} \cdot i$

9. $21 + 5i$

11. $16 - 13i$

13. -7

15. $3 - 2i$

17. $6 + 20i$

19. $-60 + 15i$

21. 24

23. $15 + 8i$

25. $-2 + \frac{2}{3}i$

27. $-9 - \sqrt{2} \cdot i$

29. $4 + \frac{\sqrt{3}}{2}i$